SOLUTION OF DIFFERENTIAL EQUATION BY METHOD OF VARIATION OF PARAMETERS

INTRODUCTION:

This method is an alternative method to find particular integral (P.I.) of linear non-homogeneous differential equation (DE) whose complementary function (C.F.) is known.

In this method, the P.I. is obtained by replacing the arbitrary constants of the C.F. with functions of x and that's why this method is known as method of variation of parameters.

METHOD:

Let us consider the general linear differential equation of second order,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X, \longrightarrow (1)$$

where P, Q and X are functions of x.

Let, $y_c = c_1 y_1 + c_2 y_2$ be the complementary function of the equation (1), where c_1, and c_2 are arbitrary constants and y_1 and y_2 are functions of x and they are independent solutions of the corresponding homogeneous equation.

i.e. $\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1 = 0$ and $\frac{d^2y_2}{dx^2} + P\frac{dy_2}{dx} + Qy_2 = 0$

[Equation (1) is homogeneous if $X=0$, otherwise it is non-homogeneous.]

Now let us assume that $y_p = uy_1 + vy_2$ be the particular integral of equation (1), where u and v are the functions of x and they are given by

$$u = -\int \frac{y_2 X}{W} dx \quad \text{and} \quad v = \int \frac{y_1 X}{W} dx$$

, where $W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

After calculating u and v, we put the values in the equation $y_p = uy_1 + vy_2$ to get the particular integral of the equation ①.

Hence the general solution of the given equation will be $y = y_c + y_p$

i.e. $y = (c_1y_1 + c_2y_2) + (uy_1 + vy_2)$

$= (u+c_1)y_1 + (v+c_2)y_2$.

APPLICATION:

Example-1: Apply the method of variation of parameters to solve,

$$\frac{d^2y}{dx^2} + 4y = \sin 2x$$

Solution: Let, $y = e^{mx}$ be a trial solution of the corresponding homogeneous equation i.e., of $\frac{d^2y}{dx^2} + 4y = 0$.

Then the auxiliary equation is

$$m^2 + 4 = 0$$

$$\Rightarrow m^2 = -4$$

$$\Rightarrow m = \pm 2i$$

$\therefore$ The complementary function of the given equation is

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

, where c_1 and c_2 are arbitrary constants.

Let, the particular integral of the given equation be $y_p = u\cos 2x + v\sin 2x$,

where, $u = -\int \frac{y_2 X}{W} dx$

and $v = \int \frac{y_1 X}{W} dx$, $W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Here $y_1 = \cos 2x$, $y_2 = \sin 2x$, $X = \sin 2x$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2(\cos^2 2x + \sin^2 2x)$$

$$= 2$$

$$\therefore\ u = -\int \frac{\sin 2x \cdot \sin 2x}{2}\,dx$$
$$= -\frac{1}{2}\int \sin^2 2x\,dx$$
$$= -\frac{1}{4}\int (1-\cos 4x)\,dx \qquad \left[\because\ 2\sin^2 2x = 1-\cos 4x\right]$$
$$= -\frac{1}{4}\left[x - \frac{\sin 4x}{4}\right]$$

and $$v = \int \frac{\cos 2x \cdot \sin 2x}{2}\,dx$$
$$= \frac{1}{2}\int \sin 2x \cdot \cos 2x\,dx$$
$$= \frac{1}{4}\int \sin 4x\,dx \qquad \left[\because\ 2\sin 2x\cos 2x = \sin 4x\right]$$
$$= \frac{1}{4}\left(-\frac{\cos 4x}{4}\right)$$
$$= -\frac{1}{16}\cos 4x.$$

$$\therefore\ y_p = uy_1 + vy_2 = -\frac{1}{4}\left(x - \frac{\sin 4x}{4}\right)\times\cos 2x + \left(-\frac{1}{16}\cos 4x\right)\times \sin 2x$$
$$= -\frac{1}{4}x\cos 2x\,\frac{\sin 4x\cos 2x}{16} - \frac{1}{16}\cos 4x \sin 2x$$
$$= -\frac{1}{4}x\cos 2x + \frac{1}{16}\left(\sin 4x\cos 2x - \cos 4x \sin 2x\right)$$
$$= -\frac{1}{4}x\cos 2x + \frac{1}{16}\sin 2x$$
$$\left[\because\ \sin ax\cos bx - \cos ax \sin bx = \sin(a-b)x\right]$$

$\therefore$ The general solution of the given differential equation is

$$y = y_c + y_p$$

i.e. $$y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{16}\sin 2x$$
$$= \left(c_1 - \frac{1}{4}x\right)\cos 2x + \left(c_2 + \frac{1}{16}\right)\sin 2x.$$

Example-2 :→ Apply the variation of parameters to solve

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 9e^x$$

Solution :→ Let, $y = e^{mx}$ be a trial solution of the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

∴ The auxiliary equation is

$$m^2 - 3m + 2 = 0$$
$$\Rightarrow m^2 - 2m - m + 2 = 0$$
$$\Rightarrow m(m-2) - 1(m-2) = 0$$
$$\Rightarrow (m-1)(m-2) = 0$$
$$\Rightarrow m = 1, 2$$

∴ The complementary function of the given equation is $y_c = c_1 e^x + c_2 e^{2x}$, where c_1 and c_2 are arbitrary constants.

Let, $y_p = ue^x + ve^{2x}$ be the particular integral of the given equation, where

$$u = -\int \frac{y_2 X}{W} dx \text{ and } v = \int \frac{y_1 X}{W} dx.$$

Here $y_1 = e^x$, $y_2 = e^{2x}$, $X = 9e^x$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$$

$$\therefore u = -\int \frac{e^{2x} \cdot 9e^x}{e^{3x}} dx = -\int 9\,dx = -9x$$

$$v = \int \frac{e^x \cdot 9e^x}{e^{3x}} dx = \int 9e^{-x} dx = -9e^{-x}$$

$$\therefore y_p = uy_1 + vy_2 = -9x \times e^x - 9e^{-x} \times e^{2x}$$
$$= -9xe^x - 9e^x = -9(x+1)e^x.$$

∴ The general solution of the given differential equation is $y = y_c + y_p$.

i.e. $y = c_1 e^x + c_2 e^{2x} - 9e^x(x+1)$.

Exercise :→

1. The C.F. of $\frac{d^2y}{dx^2}+y=\cos x$ is $A\sin x+B\cos x$. Find the P.I.

2. The C.F. of $\frac{d^2y}{dx^2}+4y=\sin 2x$ is $A\cos 2x+B\sin 2x$. Find the P.I.

3. The C.F. of $(D^2-2D+1)y=x\sin x$ is $y=(A+Bx)e^x$. Find P.I.

[**Hints**: For problem-1., 2., and 3. we only calculate the P.I.
For example, we start from here,
For Problem-1., Let us consider
$y_p=u\sin x+v\cos x$ be the P.I. of the given eqn.]

4. Apply the method of variation of parameters to solve the following equations:

i) $\frac{d^2y}{dx^2}-y=8e^x$

ii) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=65\cos 2x$

iii) $\frac{d^2y}{dx^2}+y=x\sin x$

iv) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}=e^x\sin x$

v) $\frac{d^2y}{dx^2}-y=\frac{2}{1+e^x}$

vi) $(D^2+2D+1)y=e^{-x}\log x$

vii) $(D^2-3D+2)y=xe^x+2x$
[Ans: $y=c_1e^x+c_2e^{2x}-\frac{x^2e^x}{2}-xe^{-x}+x+\frac{3}{2}$]

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Answers:

1. $y_p=\frac{1}{2}x\sin x+\frac{1}{4}\cos x$, 2. $y_p=-\frac{x}{4}\cos 2x$,
3. $y_p=\frac{1}{2}(x\cos x+\cos x-\sin x)$
4. i) $y=c_1e^x+c_2e^{-x}+(4x-2)e^x$, ii) $y=Ae^{-x}+Be^{-3x}+8\sin 2x-\cos 2x$
iii) $y=c_1\cos x+c_2\sin x+\frac{x}{2}\sin x-\frac{x^2}{4}\cos x$
iv) $y=c_1+c_2e^{2x}-\frac{1}{2}e^x\sin x$
v) $y=c_1e^x+c_2e^{-x}+e^x\log(1+e^{-x})-1-e^{-x}\log(1+e^x)$
vi) $y=c_1e^{-x}+c_2xe^{-x}+\frac{x^2}{2}\left(\frac{1}{2}-\log x\right)e^{-x}+e^{-2x}(x\log x-x)$